ON THE DYNAMIC STABILITY OF NONLINEARLY-ELASTIC THREE-LAYERED PLATES

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1. Consider a plate which is oriented with an orthogonal coordinate system α, β, γ such that the middle plane of symmetry of the plate coincides with the α, β-plane. We make the following assumptions [1,2]:

a) the normals of the plate assembly considered as a whole do not deform;

b) the material in each layer of the plate is incompressible;

c) the directions of the stress and strain tensors coincide in each layer of the plate;

d) there exists between the magnitudes of the stress and strain tensors the following relation:

$$T_i = a_i E_i - b_i E_i^{m_i} \tag{1.1}$$

Here i is the number of the layer, a_i, b_i, m_i are certain constants, and b_i can be either positive [2,3] or negative [4].

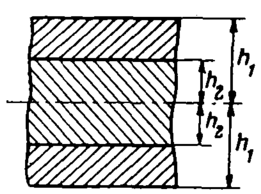

Fig. 1.

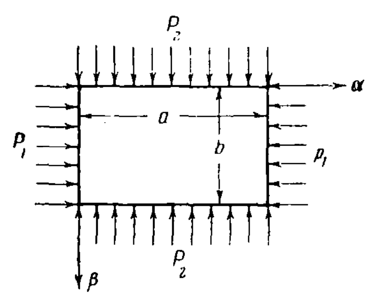

Fig. 2.

Under the above assumptions the equations for the normal displacements w have the form [1]

$$D_e \Delta^2 w - D_p \Delta^2 w - 2\frac{\partial D_p}{\partial \alpha}\left(\frac{\partial^3 w}{\partial \alpha^3} + \frac{\partial^3 w}{\partial \alpha\, \partial \beta^2}\right) - 2\frac{\partial D_p}{\partial \beta}\left(\frac{\partial^3 w}{\partial \beta^3} + \frac{\partial^3 w}{\partial \alpha^2\, \partial \beta}\right) -$$
$$- \frac{\partial^2 D_p}{\partial \alpha^2}\left(\frac{\partial^2 w}{\partial \alpha^2} + \frac{1}{2}\frac{\partial^2 w}{\partial \beta^2}\right) - \frac{\partial^2 D_p}{\partial \beta^2}\left(\frac{\partial^2 w}{\partial \beta^2} + \frac{1}{2}\frac{\partial^2 w}{\partial \alpha^2}\right) - \frac{\partial^2 D_p}{\partial \alpha\, \partial \beta}\frac{\partial^2 w}{\partial \alpha \partial \beta}$$
$$= Z(\alpha, \beta, t) - m^* \frac{\partial^2 w}{\partial t^2} \tag{1.2}$$

Here

$$D_e = \frac{8}{9}\left[a_2 h_2^3 + a_1 (h_1^3 - h_2^3)\right], \qquad D_p = \sum_{i=1}^{2} D_i P_x^{(m_i - 1)/2}$$
$$D_1 = 2^{m_1+2} 3^{-(m_1+1)/2} \frac{h_1^{m_1+2} - h_2^{m_1+2}}{m_1 + 2}, \qquad D_2 = 2^{m_2+2} 3^{-(m_2+1)/2} \frac{h_2^{m_2+2}}{m_2 + 2}$$
$$P_x = \left(\frac{\partial^2 w}{\partial \alpha^2}\right)^2 + \left(\frac{\partial^2 w}{\partial \beta_2}\right)^2 + \left(\frac{\partial^2 w}{\partial \alpha\, \partial \beta}\right)^2 + \frac{\partial^2 w}{\partial \alpha^2}\frac{\partial^2 w}{\partial \beta^2} \tag{1.3}$$
$$m^* = \frac{2}{g}\left[\gamma_2 h_2 + \gamma_1 (h_1 - h_2)\right]$$

where g is the acceleration of gravity and γ_i is the specific gravity of the material in the ith layer of the plate.

2. Let uniformly distributed compressive forces P_1 and P_2 (Fig. 2), varying with time, act in the α, β-plane of the plate. Then, as is well known [3], we have

$$\Delta Z = - P_1 \frac{\partial^2 w}{\partial \alpha^2} - P_2 \frac{\partial^2 w}{\partial \beta^2} \tag{2.1}$$

Substituting (2.1) into (1.2), we obtain the equation for the dynamic stability of a nonlinearly-elastic, three-layered plate

$$D_e \Delta^2 w - D_p \Delta^2 w - 2\frac{\partial D_p}{\partial \alpha}\left(\frac{\partial^3 w}{\partial \alpha^3} + \frac{\partial^3 w}{\partial \alpha\, \partial \beta^2}\right) - 2\frac{\partial D_p}{\partial \beta}\left(\frac{\partial^3 w}{\partial \beta^3} + \frac{\partial^3 w}{\partial \alpha^2\, \partial \beta}\right) -$$
$$- \frac{\partial^2 D_p}{\partial \alpha^2}\left(\frac{\partial^2 w}{\partial \alpha^2} + \frac{1}{2}\frac{\partial^2 w}{\partial \beta^2}\right) - \frac{\partial^2 D_p}{\partial \beta^2}\left(\frac{\partial^2 w}{\partial \beta^2} + \frac{1}{2}\frac{\partial^2 w}{\partial \alpha^2}\right) -$$
$$- \frac{\partial^2 D_p}{\partial \alpha\, \partial \beta}\frac{\partial^2 w}{\partial \alpha\, \partial \beta} + m^* \frac{\partial^2 w}{\partial t^2} + P_1(t)\frac{\partial^2 w}{\partial \alpha^2} + P_2(t)\frac{\partial^2 w}{\partial \beta^2} = 0 \tag{2.2}$$

The solution of (2.2) has the form

$$w = f(t)\, X(\alpha)\, Y(\beta) \tag{2.3}$$

where $f(t)$ is an unknown function of time and $X(\alpha)$, $Y(\beta)$ are functions which depend on a single argument and which are chosen beforehand so as to satisfy the boundary conditions that are prescribed on the edges of the rectangular contour in terms of w. For $X(\alpha)$ and $Y(\beta)$ one may take the fundamental functions for a beam.

Substituting (2.3) into (2.2) and using the Bubnov-Galerkin method to determine f, we obtain the nonlinear differential equation

$$f'' + \omega^2\Big(1 - \frac{P_1}{P_{1*}} - \frac{P_2}{P_{2*}}\Big)f - \alpha_1 |f|^{m_1-1}f - \alpha_2 |f|^{m_2-1}f = 0 \tag{2.4}$$

where

$$\omega^2 = D_e \frac{J_3}{m^* J_0}, \qquad P_{1*} = D_e \frac{J_3}{J_1}, \qquad P_{2*} = D_e \frac{J_3}{J_2}, \qquad \alpha_i = D_i \frac{J^i}{m^* J_0}$$

$$J_0 = \int_0^a\int_0^b X^2Y^2\, d\alpha\, d\beta, \qquad J_3 = \int_0^a\int_0^b (X^{\mathrm{IV}}Y + 2X''Y'' + XY^{\mathrm{IV}})\, XY\, d\alpha\, d\beta$$

$$J_1 = -\int_0^a\int_0^b X''XY^2\, d\alpha\, d\beta, \qquad J_2 = -\int_0^a\int_0^b X^2Y''Y\, d\alpha\, d\beta$$

$$J^i = \int_0^a\int_0^b K^{(m_i-1)/2}(X^{\mathrm{IV}}Y + 2X''Y'' + XY^{\mathrm{IV}}) + \frac{m_i-1}{2} K^{(m_i-3)/2} \times$$

$$\times \Big[2\frac{\partial K}{\partial \alpha}(X'''Y + X'Y'') + 2\frac{\partial K}{\partial \beta}(XY''' + X''Y') + \frac{\partial^2 K}{\partial \alpha^2}\Big(X''Y + \frac{1}{2}XY''\Big) +$$

$$+ \frac{\partial^2 K}{\partial \beta^2}\Big(XY'' + \frac{1}{2}X''Y\Big) + \frac{\partial^2 K}{\partial \alpha\, \partial \beta}X'Y'\Big] + \frac{(m_i-1)(m_i-3)}{4}K^{(m_i-5)/2} \times$$

$$\times \Big[\Big(\frac{\partial K}{\partial \alpha}\Big)^2\Big(X''Y + \frac{1}{2}XY''\Big) + \Big(\frac{\partial K}{\partial \beta}\Big)^2\Big(XY'' + \frac{1}{2}X''Y\Big) + \frac{\partial K}{\partial \alpha}\frac{\partial K}{\partial \beta}X'Y'\Big]\Big\}\, XY\, d\alpha\, d\beta$$

$$K = (X''Y)^2 + (XY'')^2 + (X'Y')^2 + X''Y\, XY'' \tag{2.5}$$

Here ω is the linear value of the natural frequency of oscillation; P_{1*} and P_{2*} are the critical values of the loads P_1 and P_2 when they are acting statically and independently.

In the particular case of $m_1 = m_2 = 3$ we have

$$f'' + \omega^2\Big(1 - \frac{P_1}{P_{1*}} - \frac{P_2}{P_{2*}}\Big)f - \alpha f^3 = 0 \tag{2.6}$$

where

$$\alpha = \frac{1}{m^* J_0}\sum_{i=1}^{2} D_i \int_0^a\int_0^b \Big[K(X^{\mathrm{IV}}Y + 2X''Y'' + XY^{\mathrm{IV}}) + 2\frac{\partial K}{\partial \alpha}(X'''Y + X'Y'') +$$

$$+ 2\frac{\partial K}{\partial \beta}(XY''' + X''Y') + \frac{\partial^2 K}{\partial \alpha^2}\Big(X''Y + \frac{1}{2}XY''\Big) +$$

$$+ \frac{\partial^2 K}{\partial \beta^2}\Big(XY'' + \frac{1}{2}X''Y\Big) + \frac{\partial^2 K}{\partial \alpha\, \partial \beta}X'Y'\Big]XY\, d\alpha\, d\beta \tag{2.7}$$

An equation analogous to (2.6) was obtained by Bolotin [3] for the homogeneous nonlinearly-elastic beam.

In the case of an infinite strip which is simply supported along its edges (Fig. 3), we set $X = \sin(\pi\alpha/a)$ and obtain from (2.5)

$$\omega^2 = \frac{D_e\lambda^4}{m^*}, \qquad P_{1*} = D_e\lambda^2, \qquad \lambda = \frac{\pi}{a}$$

$$\alpha_i = \frac{2m_i D_i \lambda^{2(m_i+1)}}{m^* a}\int_0^a [m_i \sin^{m_i+1}\lambda\alpha + (m_i - 1)\sin^{m_i-1}\lambda\alpha]\, d\alpha \tag{2.8}$$

3. Let

$$P_i = P_{i0} + P_{it}\cos\theta t \qquad (i = 1, 2) \tag{3.1}$$

Then, taking into account linear damping, Equation (2.4) may be brought to the form

$$f'' + 2\varepsilon f' + \Omega^2(1 - 2\mu\cos\theta t) f - \alpha_1 |f|^{m_1-1} f - \alpha_2 |f|^{m_2-1} f = 0 \tag{3.2}$$

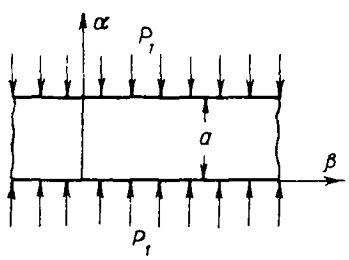

Fig. 3.

Here ϵ is the coefficient of linear damping

$$\Omega^2 = \omega^2\left(1 - \frac{P_{10}P_{2*} - P_{20}P_{1*}}{P_{1*}P_{2*}}\right)$$

$$\mu = \frac{1}{2}\,\frac{P_{1t}P_{2*} + P_{2t}P_{1*}}{P_{1*}P_{2*} - P_{10}P_{2*} - P_{20}P_{1*}} \tag{3.3}$$

It is known [3] that (3.2) can be written in a form such that its "linear part" allows of a periodic solution with the period $T = 2\pi/\theta$ or $2T$

$$f'' + 2\varepsilon_* f' + \Omega_*^2(1 - 2\mu\cos\theta t) f + V(f, f', t) = 0 \tag{3.4}$$

Here

$$\Omega_* = \Omega\frac{\theta}{\theta_*}, \qquad \varepsilon_* = \varepsilon\frac{\theta}{\theta_*} \tag{3.5}$$

$$V(f, f', t) = 2(\varepsilon - \varepsilon_*) f' + (\Omega^2 - \Omega_*^2)(1 - 2\mu\cos\theta t) f - \alpha_1 |f|^{m_1-1} f - \alpha_2 |f|^{m_2-1} f$$

The critical frequency θ_* is determined from the assumption that the initial undisturbed state is not deformed [3,5]. For example, for the boundaries of the principal region of instability we have

$$\theta_*^2 \approx 4\Omega^2 \left(1 \mp \sqrt{\mu^2 - \frac{4\varepsilon^2}{\Omega^2}}\right) \tag{3.6}$$

On the boundaries of this region, i.e. when $\theta = \theta_*$, the linear part of Equation (3.4) permits of periodic solutions, which for $\mu \ll 1$ can be approximately represented in the following form:

$$\varphi_1(t) \approx \cos\left(\frac{\theta t}{2} - \sigma\right), \qquad \varphi_2(t) \approx \sin\left(\frac{\theta t}{2} - \sigma\right) \qquad \left(\sigma \approx \frac{1}{2}\sin^{-1}\frac{\theta^3 \varepsilon_*}{4\mu\Omega_*^2}\right) \tag{3.7}$$

On the basis of the method of Mandel'shtam [3], the amplitude of the steady-state oscillations C for the boundaries of the principal region of instability can be determined, to the zeroth order, from the equation

$$\int_0^{2T} V\,[C\varphi_i(t),\, C\varphi_i'(t),\, t]\,\varphi_i(t)\,dt = 0 \tag{3.8}$$

Therefore, we obtain the nonlinear algebraic equation

$$A_1 C^{m_1} + A_2 C^{m_2} = (\Omega^2 - \Omega_*^2)(1 \mp \mu\cos 2\sigma)\,C \tag{3.9}$$

where by virtue of (3.7)

$$A_i = \frac{\alpha_i \theta}{2\pi}\int_0^{2T}\left|\cos^{m_i+1}\left(\frac{\theta t}{2} - \sigma\right)\right| dt, \quad \text{or} \quad A_i = \frac{\alpha_i \theta}{2\pi}\int_0^{2T}\left|\sin^{m_i+1}\left(\frac{\theta t}{2} - \sigma\right)\right| dt \tag{3.10}$$

We note that in the last term of Equation (3.9) the minus sign refers to the lower, and the plus sign to the upper boundary of the principal region of instability.

Examining Formulas (1.3), (2.5) and (3.10), it is easy to see that the coefficients A_i are always positive for $b_i > 0$ and negative for $b_i < 0$. We note also that the coefficients can be zero only in those cases when the corresponding layer of the plate is made of a linearly-elastic material.

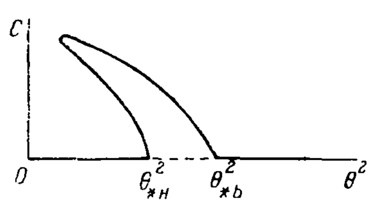

Fig. 4.

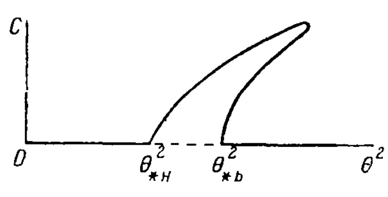

Fig. 5.

Let $A_i \geqslant 0$. In this case, on the basis of (3.9), the plot of the amplitudes of steady-state oscillations in the principal region of dynamic instability has the form shown in Fig. 5. Distortion of the oscillation in the direction of lower frequencies occurs. In the case of $A_i \leqslant 0$, the amplitude plot changes its form and the distortion of the oscillations occurs in the direction of higher frequencies (Fig. 4).

When the A_i have opposite sign, for example $A_1 > 0$, $A_2 < 0$, the first term in (3.9) has a tendency to distort the oscillations in the direction

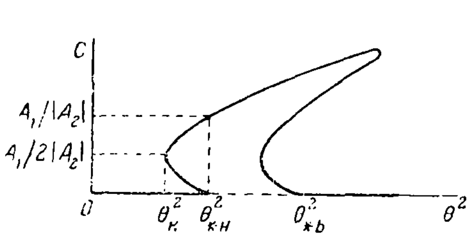

Fig. 6.

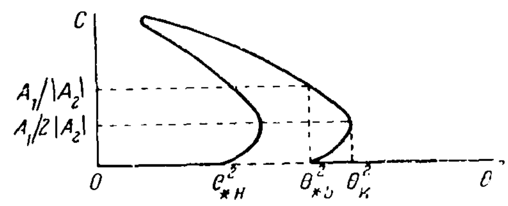

Fig. 7.

of lower frequencies, while the second term distorts the oscillations in the direction of higher frequencies. As an example we examine the case $m_1 = 2$, $m_2 = 3$. Then (3.9) has the form

$$|A_2|\, C^3 - A_1 C^2 + (\Omega^2 - \Omega_*^2)(1 \mp \mu \cos 2\tau)\, C = 0 \tag{3.11}$$

With (3.6) and (3.7) taken into account, the solution of (3.11) has the form

$$C = 0 \tag{3.12}$$

$$C = \frac{A_1}{2\,|A_2|} \pm \left(\frac{A_1^2}{4A_2^2} - \frac{\Omega^2}{|A_2|}\left(1 - \frac{\theta^2}{\theta_*^2}\right)\left(1 \mp \sqrt{\mu^2 - \frac{\theta^2\varepsilon^2}{\omega^4}}\right)\right)^{1/2} \tag{3.13}$$

The zero solution (3.12) is stable everywhere, except in the region of excitation of the linear system, where the steady-state solution (3.13) (Fig. 6) is obtained. Here the following phenomenon develops: after the plate enters resonance, the frequency begins to drop to a certain value θ_k^2, where θ_k^2 is determined from the equation

$$\frac{A_1^2}{4\,|A_2|} - \Omega^2\left(1 - \frac{\theta_k^2}{\theta_*^2}\right)\left(1 \mp \sqrt{\mu^2 - \frac{\theta_k^2\varepsilon^2}{\omega^4}}\right) = 0 \tag{3.14}$$

after which the frequency begins to rise again. The distortion of the oscillations occurs in the direction of higher frequencies.

In the case of $m_1 = 3$, $m_2 = 2$, the plot of the amplitude of steady-state oscillations changes its form (Fig. 7) and the distortion of the oscillations occurs in the direction of lower frequencies. We note further that if $m_1 = m_2$, then, depending on the moduli of the coefficients

A_i, the steady-state oscillations are established along the curve shown in Fig. 4 or Fig. 5. In this particular case, if $|A_1| = |A_2|$, the nonlinearity vanishes.

In order to complete the picture we remark that when $A_1 < 0$, $A_2 > 0$ all of the above-indicated calculations are repeated, but the plots of the steady-state amplitude dependence change their form. Depending on m_i they are represented as follows: for $m_1 = 2$, $m_2 = 3$ (Fig. 7), for $m_1 = 3$, $m_2 = 2$ (Fig. 6), for $m_1 = m_2$ (Fig. 5) or (Fig. 4).

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